On the Drach superintegrable systems.

A.V. Tsiganov

Department of Mathematical and Computational Physics, Institute of Physics,
St.Petersburg University, 198 904, St.Petersburg, Russia,
e-mail: tsiganov@mph.phys.spbu.ru

Cubic invariants for two-dimensional degenerate Hamiltonian systems are considered by using variables of separation of the associated Stäckel problems with quadratic integrals of motion. For the superintegrable Stäckel systems the cubic invariant is shown to admit new algebro-geometric representation that is far more elementary than the all the known representations in physical variables. A complete list of all known systems on the plane which admit a cubic invariant is discussed.

1 Introduction

In 1935 Jules Drach applied direct method for search of the integrable Hamiltonian systems of two degrees of freedom, which admit a cubic second integral [2]. Recall, the direct approach leads to a complicated set of nonlinear equations, whose nonlinearity has no *a priory* restriction. We can attempt to solve these second invariant differential equations using various simplifying assumptions. The Drach ansatz for the Hamilton function H and for the second cubic invariant K

$$
\begin{aligned}
H &= p_x p_y + U(x,y)\,, \\
K &= 6w(x,y)\left(\frac{\partial H}{\partial x}\,p_y - p_x\,\frac{\partial H}{\partial y}\right) - P(p_x,p_y,x,y)
\end{aligned}
\tag{1.1}
$$

yields ten new integrable systems. Recall, for the the fixed polynomial $P(p_x,p_y,x,y)$ the potential $U(x,y)$ and function $w(x,y)$ be solution of some differential equations [2].

For the other natural Hamilton functions

$$
H = p_x^2 + p_y^2 + V(x,y) \tag{1.2}
$$

the similar approach has been used by Fokas and Lagerstrom [3], by Holt [5] and by Thompson [9]. Note, that the some Drach results have been rediscovered in these papers.

The most complete classifications of known results was later brought together by Hietarinta in 1987 [4]. Recently [6], two-dimensional hamiltonian systems with the cubic integrals were investigated using the Jacobi change of the time. In [6] a complete list of all known systems was extended in comparison with [4].

The aim of this note is to study the Drach systems and some other degenerate systems on the plane with the cubic in momenta integrals of motion. We prove that eight Drach hamiltonians belong to the Stackel family of integrals [8] and, moreover, seven of them are degenerate systems.

Recall, the system is called superintegrable or degenerate if the Hamilton function H is in the involution with two integrals of motion I and K, such that

$$
\{H,I\} = \{H,K\} = 0\,, \qquad \{I,K\} = J(H,I,K)\,. \tag{1.3}
$$

Initial integrals and the constant of motion $J(H,I,K)$ are generators of the polynomial associative algebra [1, 7], whose defining relations are polynomials of certain order in generators.

Below we shall consider two-dimensional systems with two quadratic integrals of motion $I_1 = H,\ I_2 = I$ and one qubic integral K. So, by the Bernard-Darboux theorem [17] the system with integrals $I_1,\ I_2$ belong to the Stäckel family of integrable systems [8]. Therefore, let us begin with remaining of some necessary results about the Stäckel systems [8].

2 The Stäckel systems

The systems associated with the name of Stäckel [8] are holonomic systems on the phase space $\mathbb{R}^{2n}$ equipped with the canonical variables $\{p_j, q_j\}_{j=1}^n$. The nondegenerate $n\times n$ Stäckel matrix $\mathbf{S}$, with entries s_{kj} depending only on q_j

$$\det \mathbf{S}\neq 0\,,\qquad \frac{\partial s_{kj}}{\partial q_m}=0\,,\quad j\neq m$$

defines n functionally independent integrals of motion

$$I_k=\sum_{j=1}^n c_{jk}\left(p_j^2+U_j\right)\,,\qquad c_{jk}=\frac{\mathfrak{s}^{kj}}{\det\mathbf{S}}\,, \tag{2.1}$$

which are quadratic in momenta. Here $\mathbf{C}=[c_{jk}]$ denotes inverse matrix to $\mathbf{S}$ and $\mathfrak{s}^{kj}$ be cofactor of the element s_{kj}. The common level surface of these integrals

$$M_\alpha=\left\{z\in\mathbb{R}^{2n}: I_k(z)=\alpha_k\,,\ k=1,\ldots,n\right\}$$

is diffeomorphic to the n-dimensional real torus and one immediately gets

$$p_j^2=\left(\frac{\partial\mathcal{S}}{\partial q_j}\right)^2=\sum_{i=1}^n\alpha_i s_{ij}(q_j)-U_j(q_j)\,. \tag{2.2}$$

Here $\mathcal{S}(q_1\ \ldots,q_n)$ is a reduced action function [8]. For the rational entries of $\mathbf{S}$ and rational potentials $U_j(q_j)$ one gets

$$p_j^2=\frac{\prod_{i=1}^k(q_j-e_i)}{\varphi_j^2(q_j)}\,, \tag{2.3}$$

where e_i are constants of motion and functions $\varphi_j(q_j)$ depend on coordinate q_j and numerical constants [11]. The Riemann surfaces (2.3) are isomorphic to the canonical hyperelliptic curves

$$\mathcal{C}_j:\quad \mu_j^2=\prod_{i=1}^k(\lambda-e_i)\,,\qquad \mu_j=\varphi(q_j)\,p_j\,, \tag{2.4}$$

where the senior degree k of polynomial fixes the genus $g_j=[(k-1)/2]$ of the algebraic curve $\mathcal{C}_j$. Considered together, these curves determine an n-dimensional Lagrangian submanifold in $\mathbb{R}^{2n}$

$$\mathcal{C}^{(n)}:\qquad \mathcal{C}_1(p_1,q_1)\times\mathcal{C}_2(p_2,q_2)\times\cdots\times\mathcal{C}_n(p_n,q_n)\,.$$

The Abel transformation linearizes equations of motion on $\mathcal{C}^{(n)}$ by using first kind abelian differentials on the corresponding algebraic curves [12]. The basis of first kind abelian differentials is uniquely related to the Stäckel matrix $\mathbf{S}$ [11, 12].

Now let us turn to the superintegrable or degenerate systems in the classical mechanics. One of the main examples of the two-dimensional superintegrable systems is the isotropic harmonic oscillator, which has many common properties with the Drach degenerate systems. Recall, for the oscillator the Hamilton function and the second integral of motion look like

$$H=p_1^2+p_2^2+q_1^2+q_2^2\,,\qquad I=p_1^2+q_1^2-p_2^2-q_2^2\,.$$

Obviously, the angular momentum

$$K=q_1p_2-p_1q_2=\frac12\left(p_1\,\frac{dp_2}{dt}-\frac{dp_1}{dt}\,p_2\right)\,. \tag{2.5}$$

is the third integral of motion. Two pairs of quadratic integrals $I_1=H,\ I_2=I$ and $\tilde I_1=H,\ \tilde I_2=K^2$ are associated with the following Stäckel matrices

$$\mathbf{S}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\,,\quad\text{and}\quad \tilde{\mathbf{S}}=\begin{pmatrix}1&0\\r^{-2}&1\end{pmatrix}\,,\quad r^2=x^2+y^2$$

respectively. So, the corresponding equations of motion may be separated in the different curvilinear coordinate systems.

For this degenerate Stäckel system and for all other known superintegrable Stäckel systems with quadratic integrals of motion the number degree of freedom $n > g$ is always more then sum of genuses g_j of the corresponding algebraic curves. In this case the number of independent first kind abelian differentials be insufficient for the inversion of the Abel-Jacobi map on $\mathcal{C}^{(n)}$.

To construct inversion of this map for the degenerate systems one has to complete a given basis of the differentials to the set of n differentials. We have some freedom in a choice of complimentary differentials and, therefore, we can associate the different Stäckel matrices to one given Hamilton function [12]. By using first kind abelian differentials one gets superintegrable Stäckel system with quadratic integrals only. Of course, we can try to add the second and third kind abelian differentials, but we do not know such examples.

Below we prove that for all the known superintegrable systems with a qubic integral K the number degree of freedom $n = 2$ is more then sum $g = g_1 + g_2 = 0$ of genuses $g_j = 0$ of the associated Riemann surfaces (2.4) too. The corresponding dynamics is splitting on two spheres

$$\mathcal{C}_{1,2}: \qquad \mu^2 = \alpha_{1,2}\lambda^2 + \beta_{1,2}\lambda + \gamma_{1,2}\,, \qquad g_{1,2} = 0\,, \tag{2.6}$$

where α_j, β_j and γ_j be the constants of motion.

In variables $\mu_{1,2}$ (2.4) the additional cubic integral of motion for all the degenerate Drach systems looks like

$$K = \frac{\det \mathbf{S}}{s_{21}s_{22}} \left(\mu_1 \frac{d\mu_2}{dt} - \frac{d\mu_1}{dt}\,\mu_2 \right) . \tag{2.7}$$

This generalized "angular momentum" gives rise to the first order integrals (2.5) or the third order polynomials in momenta depending on the Stäckel matrices $\mathbf{S}$ and potentials U_j. In our case it will be qubic integral, which coincides with the initial Drach integral up to numerical factor.

To consider nonlinear algebra of integrals of motion for the Drach systems we shall introduce new generators $\{N, a, a^\dagger\}$ instead of the two quadratic integrals $I_1 = H,\ I_2 = I$, one qubic integrals K and the constant of motion J (1.3). Similar to oscillator these new generators have the following properties

$$\begin{aligned} &\{N, a\} = a\,, && \{N, a^\dagger\} = -a^\dagger\,,\\ &\{a, a^\dagger\} = \Phi(I_1, I_2)\,, && a\,a^\dagger = \Psi(I_1, I_2)\,. \end{aligned} \tag{2.8}$$

Here generator $N(I_1, I_2)$, functions $\Psi(I_1, I_2)$ and $\Phi(I_1, I_2)$ depend on the quadratic Stäckel integrals only. Two other generators a and $a^\dagger$ be functions on the all three constants of motion I_1, I_2, K, such that

$$K = \rho(I_1, I_2)\,(a - a^\dagger)\,, \qquad J(I_1, I_2, K) = \frac{a + a^\dagger}{2}\,.$$

The relations (2.8) remind the deformed oscillator algebra, which is widely used for the superintegrable systems with quadratic integrals of motion [1, 7]. However, instead of the usual quadratic algebra of integrals we shall get more complicated algebras of integrals.

3 The Drach systems

Let us reproduce corrected Drach results in his notations

$$\begin{aligned} (a) \qquad U &= \frac{\alpha}{xy} + \beta x^{r_1} y^{r_2} + \gamma x^{r_2} y^{r_1}\,, \quad \text{where} \quad r_j^2 + 3r_j + 3 = 0\,, \\ P &= (xp_x - p_y y)^3\,, \qquad w = x^2y^2/2\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned} (b) \qquad U &= \frac{\alpha}{\sqrt{xy}} + \frac{\beta}{(y-\mu x)^2} + \frac{\gamma\,(y+\mu x)}{\sqrt{xy}\,(y-\mu x)^2}\,, \\ P &= 3(xp_x - p_y y)^2(p_x + \mu p_y)\,, \qquad w = xy(y - \mu x)\,, \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
(c) \qquad U &= \alpha\,xy + \frac{\beta}{(y-ax)^2} + \frac{\gamma}{(y+ax)^2}\,, &(3.3)\\
P &= 3(xp_x - p_y y)(p_x^2 - a^2 p_y^2)\,, \qquad w = (y^2 - a^2x^2)/2\,,
\end{aligned}
$$

$$
\begin{aligned}
(d) \qquad U &= \frac{\alpha}{\sqrt{y(x-a)}} + \frac{\beta}{\sqrt{y(x+a)}} + \frac{\gamma x}{\sqrt{x^2-a^2}}\,, &(3.4)\\
P &= 3p_y\,\left[(xp_x - p_y y)^2 - a^2 p_x^2\right]\,, \qquad w = -y(x^2-a^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
(e) \qquad U &= \frac{\alpha}{\sqrt{xy}} + \frac{\beta}{\sqrt{x}} + \frac{\gamma}{\sqrt{y}}\,, &(3.5)\\
P &= 3p_y p_x\,(xp_x - p_y y)\,, \qquad w = -2xy\,,
\end{aligned}
$$

$$
\begin{aligned}
(f) \qquad U &= \alpha\,xy + \beta y\frac{2x^2+c}{\sqrt{x^2+c}} + \frac{\gamma x}{\sqrt{x^2+c}}\,, &(3.6)\\
P &= 3p_y^2\,(xp_x - yp_y)\,, \qquad w = (x^2+c)/2\,,\\
(g) \qquad U &= \frac{\alpha}{(y+3mx)^2} + \beta(y-3mx) + \gamma(y-mx)(y-9mx)\,, &(3.7)\\
P &= (p_x + 3mp_y)^2\,(p_x - 3mp_y)\,, \qquad w = -m(y+3mx)\,,
\end{aligned}
$$

$$
\begin{aligned}
(h) \qquad U &= (y + \frac{mx}{3})^{-2/3}\,\left[\alpha + \beta\,(y - mx/3) + \gamma\,(y^2 - \frac{14mxy}{3} + \frac{m^2x^2}{9})\right]\,, &(3.8)\\
P &= (p_x - \frac{mp_y}{3})\left(p_x^2 + \frac{10mp_xp_y}{3} + \frac{m^2p_y^2}{9}\right)\,, \qquad w = -m(y + \frac{mx}{3})\,,
\end{aligned}
$$

$$
\begin{aligned}
(k) \qquad U &= \alpha y^{-1/2} + \beta x y^{-1/2} + \gamma x\,, &(3.9)\\
P &= 3p_x^2 p_y\,, \qquad w = -y\,,
\end{aligned}
$$

$$
\begin{aligned}
(l) \qquad U &= \alpha\left(y - \frac{\rho x}{3}\right) + \beta x^{-1/2} + \gamma x^{-1/2}(y - \rho x)\,, &(3.10)\\
P &= 3p_x p_y^2 + \rho p_y^3\,, \qquad w = x\,.
\end{aligned}
$$

Here α, β, γ, μ, ρ, a, c, and m be arbitrary parameters. In compare with [2] we corrected function w in the case (g) (3.7) and revised potential U in the case (k) (3.9). Namely this corrected Hamiltonian is in the involution with the initial Drach cubic integral K (1.1).

With an exception of three cases (a) (3.1), (h) (3.8) and (l) (3.10), other Drach systems are degenerate or superintegrable Stäckel systems. The separation variables associated with the pair of quadratic integrals $\{I_1 = H, I_2\}$ are the Stäckel variables. Equations of motion may be integrated in quadratures [11], but these quadratures depend on the value of quadratic integral I_2. Thus, instead of the solution of initial Drach problem related to integrals $\{H, K\}$ we shall solve the associated problem with quadratic integrals $\{H, I_2\}$.

In the case (h) (3.8) we also have the Stäckel systems [13]. Only in this case (h) (3.8) dynamics is splitting on two tori and the number degrees of freedom is equal to the sum of genuses $g = n = 2$, such that the corresponding system is non-degenerate.

In the case (l) (3.10) the Hamilton function coincides with the hamiltonian of the previous Stäckel system (3.9) at $\rho = 0$. Here we shall not consider this generalized Stäckel system at $\rho \neq 0$.

Below we shall consider the Drach integrals (1.1) up to linear transformations of the coordinates and a rescaling of these integrals. It allows us to remove some parameters in the Hamilton functions without loss

of generality. To associate the degenerate Drach hamiltonians with the Stäckel matrices $\mathbf{S}$ we can join these systems into the four pairs of the systems with a common matrices $\mathbf{S}$.

3.1 Case (a)

In our previous paper [14], the first Drach system (3.1) has been related to the three-particle periodic Toda lattice in the center-of-mass frame. Namely, after canonical change of the time $t = q_{n+1}$ and the Hamiltonian $H = p_{n+1}$ at the extended phase space

$$d\widetilde{t} = (xy)^{-1} \cdot dt \,, \qquad \widetilde{H} = xy \cdot (H + \delta)\,,$$

and after further canonical transformation of other variables

$$x = e^{\frac{q_1+iq_2}{2}}\,, \quad p_x = (p_1 - ip_2)e^{-\frac{q_1+iq_2}{2}}\,, \qquad y = e^{\frac{q_1-iq_2}{2}}\,, \quad p_y = (p_1 + ip_2)e^{-\frac{q_1-iq_2}{2}}\,,$$

the Hamilton function (3.1) becomes

$$\widetilde{H} = p_1^2 + p_2^2 + \beta e^{-\frac{1}{2}q_1 - \frac{\sqrt{3}}{2}q_2} + \gamma e^{-\frac{1}{2}q_1 + \frac{\sqrt{3}}{2}q_2} + \delta e^{q_1} + \alpha\,.$$

It is the Hamiltonian of the tree-particle periodic Toda lattice in the center-of-mass frame. The separation variables survive at the change of the time. Thus, for the first Drach system we can separate variables and integrate equations of motions in quadratures repeating the calculations for the Toda chain [15].

Later in [9] Thompson considered this system too. In fact, after point transformation

$$x = r\, e^{i\phi}\,, \quad p_x = \frac{e^{-i\phi}}{2}\left(p_r - ip_\phi\, r^{-1}\right)\,, \qquad y = r\, e^{-i\phi}\,, \quad p_y = \frac{e^{i\phi}}{2}\left(p_r + ip_\phi\, r^{-1}\right)$$

the Drach hamiltonian H (1.1) looks like

$$H = p_r^2 + \frac{p_\phi^2}{r^2} + U(r,\phi)\,,$$

up to numerical factor. Namely this Hamilton function was studied in [9] and [6]. The special substitution of the potential $U(r,\phi)$ into the Drach equations leads to the following equation

$$U(r,\phi) = \frac{f(\phi) + f''(\phi)}{r^3}\,, \quad \Rightarrow \quad f'''\, f'' - 2f''\, f' - 3f' f = 0\,,$$

introduced in [6]. Of course, the same equation follows from the functional equation on the Toda potential [4].

3.2 Cases (b) and (e)

Put $\mu = 1$ in (3.2). Let us introduce the Stäckel matrix

$$\mathbf{S}_{be} = \begin{pmatrix} q_1^2 & q_2^2 \\ 1 & 1 \end{pmatrix}\,, \tag{3.11}$$

and take the following potentials

$$(b) \qquad U_1 = 2\alpha - \frac{\beta - 2\gamma}{q_1^2}, \qquad U_2 = -2\alpha - \frac{\beta + 2\gamma}{q_2^2},$$

$$(e) \qquad U_1 = 2\alpha + 2(\beta + \gamma)q_1, \qquad U_2 = -2\alpha - 2(\beta - \gamma)q_2.$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H for the Drach systems (3.2) and (3.5), after the following canonical point transformation

$$x = \frac{(q_1 - q_2)^2}{4}, \quad p_x = \frac{p_1 - p_2}{q_1 - q_2}, \qquad y = \frac{(q_1 + q_2)^2}{4}, \quad p_y = \frac{p_1 + p_2}{q_1 + q_2}.$$

The second integrals of motion I_2 (2.1) are second order polynomials in momenta. The third independent integrals of motion K are defined by (2.7), where

$$(b)\quad \mu_1=q_1p_1,\quad \mu_2=q_2p_2,\qquad (e)\quad \mu_1=p_1,\quad \mu_2=p_2\,.$$

From the above definitions we can introduce generators of the nonlinear algebra of integrals (2.8) and verify properties of this algebra

$$(b)\qquad N=\frac{I_2}{4\sqrt{H}}\,,\qquad a=J+4\sqrt{H}\,K\,,\qquad a^\dagger=J-4\sqrt{H}\,K\,,$$

$$a\,a^\dagger=16\left(4H\,(\beta+2\gamma)-(2\alpha+I_2)^2\right)\left(4H\,(\beta-2\gamma)-(2\alpha-I_2)^2\right)\,,$$

$$\{a,a^\dagger\}=-256\sqrt{H}\,\left(I_2\,(I_2-2\alpha)(I_2+2\alpha)-4H\,(\beta I_2-4\alpha\gamma)\right)\,,$$

and

$$(e)\qquad N=\frac{I_2}{2\sqrt{H}}\,,\qquad a=J+2\sqrt{H}\,K\,,\qquad a^\dagger=J-2\sqrt{H}\,K\,,$$

$$a\,a^\dagger=-16\left(H\,(2\alpha+I_2)-(\beta-\gamma)^2\right)\left(H\,(2\alpha-I_2)+(\beta+\gamma)^2\right)\,,$$

$$\{a,a^\dagger\}=-64H^{3/2}\,(I_2\,H-\beta^2-\gamma^2)\,.$$

3.3 Cases (c) and (g)

Put $a=1$ in (3.3) and $m=1/3$ in (3.7). Let us introduce the Stäckel matrix

$$\mathbf{S}_{cg}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ 1 & 1\end{pmatrix}\,,\qquad (3.12)$$

and take the following potentials

$$\begin{aligned}&(c)\qquad U_1=\frac{\alpha q_1^2}{4}+\frac{\gamma}{q_1^2},\qquad U_2=\frac{\alpha q_2}{4}-\frac{\beta}{q_2^2},\\ &(g)\qquad U_1=-\frac{\gamma q_1^2}{3}+\frac{\alpha}{q_1^2},\qquad U_2=-\frac{4\gamma q_2^2}{3}-\beta q_2.\end{aligned}\qquad (3.13)$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (3.3) and (3.7), after the following canonical point transformation

$$x=\frac{q_1-q_2}{2},\quad p_x=p_1-p_2,\qquad y=\frac{q_1+q_2}{2},\quad p_y=p_1+p_2.$$

The second integrals of motion I_2 (2.1) are the second order polynomials in momenta. The third independent integrals K are defined by (2.7), where

$$(c)\quad \mu_1=q_1p_1\quad \mu_2=q_2p_2\,,\qquad (g)\quad \mu_1=q_1p_1\,,\quad \mu_2=p_2\,.$$

Generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$(c)\qquad N=\frac{I_2}{2\sqrt{-\alpha}}\,,\qquad a=J+2\sqrt{-\alpha}\,K\,,\qquad a^\dagger=J-2\sqrt{-\alpha}\,K\,,$$

$$a\,a^\dagger=\left(H^2+4H\,I_2+4I_2^2-4\alpha\gamma\right)\left(H^2-4H\,I_2+4I_2^2+4\alpha\beta\right)\,,$$

$$\{a,a^\dagger\}=-32\sqrt{a}\,\left(I_2\,(H-2I_2\,)(H+2I_2\,)-\alpha\beta(H+2I_2\,)-\alpha\gamma(H-2I_2\,)\right)\,,$$

and

$$(g) \qquad N=\frac{I_2}{4}\sqrt{\frac{3}{\gamma}}\,,\qquad a=J+4\sqrt{\frac{\gamma}{3}}\,K\,,\qquad a^\dagger=J-4\sqrt{\frac{\gamma}{3}}\,K\,,$$

$$a\,a^\dagger=1/9\left(8\gamma H-16\gamma I_2+3\beta^2\right)\left(3H^2+12H\,I_2+12I_2^2+16\alpha\gamma\right)\,,$$

$$\{a,a^\dagger\}=64\left(\frac{\gamma}{3}\right)^{3/2}\left(\frac{(2I_2+H)(4\gamma H-24\gamma I_2+3b^2)}{4\gamma}-\frac{16\alpha\gamma}{3}\right)\,.$$

3.4 Cases (d) and (f)

Put $a=1$ in (3.4) and $c=1$ in (3.6). Let us introduce two the Stäckel matrices

$$\mathbf{S}_d==\begin{pmatrix}1 & 1\\ \\ \dfrac{1}{q_1^2} & \dfrac{1}{q_2^2}\end{pmatrix},\qquad \mathbf{S}_f=\begin{pmatrix}\dfrac{1}{q_1} & \dfrac{1}{q_2}\\ \\ \dfrac{1}{q_1^2} & \dfrac{1}{q_2^2}\end{pmatrix} \tag{3.14}$$

and takes the following potentials

$$(d) \qquad U_1=2\gamma+\frac{2\sqrt{2}(\alpha+\beta)}{q_1},\qquad U_2=-2\gamma-\frac{2\sqrt{2}(\alpha-\beta)}{q_2},$$

$$(f) \qquad U_1=\frac{\gamma}{2q_1}+\frac{(\alpha+2\beta)}{4},\qquad U_2=\frac{\gamma}{2q_2}+\frac{(\alpha-2\beta)}{4}.$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (3.4) and (3.6) up to numerical factor, after the following explicit canonical transformations

$$(d) \qquad x=\frac{q_1^2+q_2^2}{2q_1q_2},\quad p_x=\frac{(p_1q_1-p_2q_2)q_1q_2}{q_1^2-q_2^2},\qquad y=q_1q_2,\quad p_y=\frac{p_1q_1+p_2q_2}{2q_1q_2},$$

$$(f) \qquad x=\frac{q_1-q_2}{2\sqrt{q_1q_2}},\quad p_x=\frac{2(p_1q_1-p_2q_2)\sqrt{q_1q_2}}{q_1+q_2},\qquad y=\sqrt{q_1q_2},\quad p_y=\frac{p_1q_1+p_2q_2}{\sqrt{q_1q_2}}.$$

The second integrals of motion I_2 (2.1) are the quadratic polynomials in momenta. The third independent integrals K are defined by (2.7), where for the both systems one gets

$$\mu_1=q_1p_1\,,\qquad \mu_2=q_2p_2\,.$$

Generators of the nonlinear algebra of integrals (2.8) are given by

$$N=\sqrt{I_2}\,,\qquad a=J+2\sqrt{I_2}\,K\,,\qquad a^\dagger=J-2\sqrt{I_2}\,K\,,$$

which have the following properties

$$(d) \qquad a\,a^\dagger=16\left(I_2\,(2\gamma-H)+2(\alpha+\beta)^2\right)\left(I_2\,(2\gamma-H)-2(\alpha-\beta)^2\right)\,,$$

$$\{a,a^\dagger\}=64\sqrt{I_2}\left(I_2\,(2\gamma-H)(2\gamma+H)+2(\alpha^2+\beta^2)H+8\alpha\beta\gamma\right)\,,$$

$$(f) \qquad a\,a^\dagger=\frac{1}{16}\left(4I_2\,(\alpha+2\beta)+(\gamma-2H)^2\right)\left(4I_2\,(\alpha-2\beta)+(\gamma+2H)^2\right)\,,$$

$$\{a,a^\dagger\}=-4\sqrt{I_2}\left((2\beta-\alpha)(2\beta+\alpha)I_2+\alpha H^2+2\beta\gamma H+\frac{\alpha\gamma^2}{4}\right)\,.$$

3.5 Cases (h) and (k)

Put $m = 3$ in (3.8). Let us introduce two the Stäckel matrices

$$\mathbf{S}_h == \begin{pmatrix} q_1 & q_2 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{S}_k = \begin{pmatrix} q_1 & -q_2 \\ 1 & 1 \end{pmatrix} \tag{3.15}$$

and take the following potentials

$$(h) \qquad U_1 = \frac{\alpha\gamma - \dfrac{\beta^2}{16}}{4\gamma} - \frac{2\gamma\, q_1^3}{9}, \qquad U_2 = \frac{\alpha\gamma - \dfrac{\beta^2}{16}}{4\gamma} - \frac{2\gamma\, q_2^3}{9}$$

$$(k) \qquad U_1 = \alpha + \beta q_1 + \frac{\gamma q_1^2}{2}, \qquad U_2 = -\alpha + \beta q_2 + \frac{\gamma q_2^2}{2}.$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (3.8) and (3.9) up to numerical factor, after the following explicit canonical transformations

$$(h) \qquad x = \frac{p_2 - p_1}{4\sqrt{\gamma}} + \frac{(3q_1 + 3q_2)^{3/2}}{54} + \frac{\beta}{16\gamma}, \qquad p_x = 3\frac{p_1 + p_2}{\sqrt{3q_1 + 3q_2}} + \sqrt{\gamma}(q_1 - q_2),$$

$$y = -\frac{p_2 - p_1}{4\sqrt{\gamma}} + \frac{(3q_1 + 3q_2)^{3/2}}{54} - \frac{\beta}{16\gamma}, \qquad p_y = 3\frac{p_1 + p_2}{\sqrt{3q_1 + 3q_2}} - \sqrt{\gamma}(q_1 - q_2),$$

and

$$(k) \qquad x = \frac{q_1 - q_2}{2}, \quad p_x = p_1 - p_2, \qquad y = \frac{(q_1 + q_2)^2}{4}, \quad p_y = \frac{p_1 + p_2}{q_1 + q_2}.$$

Note, in the case (h) (3.8) we used non-point canonical transformation in contrast with other Drach systems.

In the last case (k) (3.9) integral of motion I_2 (2.1) is the second order polynomial in momenta. The third independent integral K is defined by (2.7), where

$$\mu_1 = p_1 \quad \text{and} \quad \mu_2 = p_2.$$

Generators and defining relations of nonlinear algebra of integrals (2.8) look like

$$(k) \qquad N = \frac{I_2}{\sqrt{-2\gamma}}, \qquad a = J + \sqrt{-2\gamma}\, K, \qquad a^\dagger = J - \sqrt{-2\gamma}\, K,$$

$$a\, a^\dagger = \left(2\gamma\,(I_2 + \alpha) + (H + \beta)^2\right)\left(2\gamma\,(I_2 - \alpha) + (H - \beta)^2\right),$$

$$\{a, a^\dagger\} = -4\gamma\sqrt{-2c}\,\left(H^2 + 2\gamma I_2 + \beta^2\right).$$

In the case (h) (3.8) the second integral of motion I_2 (2.1) is the second order polynomial in momenta $\{p_1, p_2\}$. However, after the non-point transformation of variables this integral I_2 becomes the qubic in momenta $\{p_x, p_y\}$ Drach integral K (3.8). The corresponding dynamics is splitting on two tori and the third order polynomial (2.7) does not commute with the Hamilton function. Later this system has been rediscovered by Holt [5].

4 Other degenerate systems on the plane with a qubic integral of motion

In this section we consider the Stäckel systems on the plane with a qubic integral of motion defined by the following Hamilton function

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + V(x, y).$$

As above, the corresponding qubic integral will be written at the Drach form (1.1).

On the plane we know four orthogonal systems of coordinates: elliptic, parabolic, polar and cartesian. Thus, we reproduce all the known results [4, 6] in correspondence with the type of the associated Stäckel matrix [11, 12].

The systems whose Hamilton functions separable in cartesian coordinates:

$$(A) \qquad V = \alpha\,(4x^2+y^2)+\beta\,x+\frac{\gamma}{y^2}\,, \tag{4.1}$$

$$P = p_x\,p_y^2\,, \qquad w=\frac{y}{6}\,, \tag{4.2}$$

$$(B) \qquad V = \alpha\,(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}\,, \tag{4.3}$$

$$P = (x\,p_y-y\,p_x)\,p_x\,p_y\,, \qquad w=\frac{xy}{6}\,,$$

$$(C) \qquad V = \alpha\,(x^2+y^2)+\beta\,\frac{xy}{(x^2-y^2)^2}\,, \tag{4.4}$$

$$P = (x\,p_y-y\,p_x)\,\left(p_x^2-p_y^2\right)\,, \qquad w=\frac{x^2-y^2}{6}\,,$$

$$(D) \qquad V = \alpha(9x^2+y^2)\,, \tag{4.5}$$

$$P = (x\,p_y-p_x\,y)\,p_y^2\,, \qquad w=-\frac{y^2}{18}\,,$$

The systems whose Hamilton functions separable in parabolic coordinates:

$$(F) \qquad V = (\alpha+\frac{\beta}{r+x}+\frac{\gamma}{r-x})\,r^{-1}\,, \quad r=\sqrt{x^2+y^2}\,, \tag{4.6}$$

$$P = (x\,p_y-p_x\,y)^2\,p_x\,, \qquad w=\frac{y\,r^2}{12}\,,$$

$$(G) \qquad V = (\alpha+\frac{\beta\,x}{y^2})\,r^{-1}\,, \tag{4.7}$$

$$P = (x\,p_y-p_x\,y)^2\,p_x\,, \qquad w=\frac{y\,r^2}{12}\,,$$

$$(H) \qquad V = (\alpha+\beta\,\sqrt{r+x}+\gamma\,\sqrt{r-x})\,r^{-1}\,, \tag{4.8}$$

$$P = (x\,p_y-p_x\,y)\,\left(2p_x^2+2p_y^2-\frac{\beta}{\sqrt{r+x}}-\frac{\gamma}{\sqrt{r-x}}\right)\,, \qquad w=-\frac{r^2}{6}\,,$$

One system with the Hamilton function separable in polar coordinates:

$$(I) \qquad V = \alpha+\frac{\beta}{\sqrt{x^2+y^2}}+\frac{\rho}{(\delta\,x+\gamma\,y)^2}+\frac{\gamma\,x-\delta\,y}{\sqrt{x^2+y^2}\,(\delta\,x+\gamma\,y)^2}\,, \tag{4.9}$$

$$P = (p_x\,y-p_y\,x)(\gamma\,p_x-\delta\,p_y)\,, \qquad w=\frac{(x^2+y^2)\,(\delta\,x+\gamma\,y)}{12}\,.$$

It is new superintegrable system with a cubic integral of motion, which is a deformation of the degenerate kepler model. An application of the direct method [3, 5, 4] or the Jacobi method [6] does not allows us to obtain this system (4.9).

Three exceptional systems whose qubic integral of motion K we can not rewrite in the "generalized angular momentum" form (2.7):

$$(K) \qquad V = \alpha\sqrt{x} \pm \beta\sqrt{y}\,, \qquad P = \frac{\beta}{\alpha}\,p_x^3 \mp \frac{\alpha}{\beta}\,p_y^3\,, \qquad w = \sqrt{xy}\,, \tag{4.10}$$

$$(L) \qquad V = \alpha\left(\sqrt{x} + \beta\,y\right)\,, \qquad P = p_x^3\,, \qquad w = -\frac{\sqrt{x}}{2\beta}\,, \tag{4.11}$$

$$\begin{aligned}(M) \qquad V &= f'(\phi)\,r^{-2}\,, \qquad K = p_\phi^2\,\left(\cos\phi\,p_r - \sin\phi\,r^{-1}\,p_\phi\right) + \\ &+ \left(2f'(\phi)\cos\phi - f(\phi)\sin(\phi)\right)\,p_r + \left(3f'(\phi)\sin\phi + f(\phi)\cos\phi\right)r^{-1}p_\phi\,.\end{aligned} \tag{4.12}$$

At the case (M) (4.12) we used the standard polar coordinates $\{r, p_r, \phi, p_\phi\}$ and the function $f(\phi)$ has to satisfy the following equation

$$f''\,\left(3f'\sin\phi + f\cos\phi\right) + 2f'\,\left(2f'\cos\phi - f\sin\phi\right) = 0\,.$$

At these exceptional cases the Hamilton functions (4.10,4.11,4.12) are separable at the cartesian and polar coordinates, respectively. However, the Stäckel potentials $U_{1,2}$ are not polynomials in variables of separation.

4.1 Cartesian coordinates, cases A-D

Let us introduce the Stäckel matrix

$$\mathbf{S}_{A-D} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}\,, \tag{4.13}$$

and take the following potentials

$$\begin{array}{lll} (A) & U_1 = 8\alpha\,q_1^2 + 2\beta\,q_1, & U_2 = 2\alpha\,q_2^2 + \dfrac{2\gamma}{q_2^2}, \\ (B) & U_1 = 2\alpha\,q_1^2 + \dfrac{2\beta}{q_1^2}, & U_2 = 2\alpha\,q_2^2 + \dfrac{2\gamma}{q_2^2}, \\ (C) & U_1 = \dfrac{\alpha\,q_1^2}{2} - \dfrac{\beta}{4q_1^2}, & U_2 = \dfrac{\alpha\,q_2^2}{2} + \dfrac{\beta}{4q_2^2}, \\ (D) & U_1 = 18\alpha\,q_1^2, & U_2 = 2\alpha\,q_2^2. \end{array} \tag{4.14}$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (4.1,4.3,4.5) and (4.4) if

$$(A-B, D) \qquad x = q_1, \qquad y = q_2$$

or after the following canonical transformation

$$(C) \qquad x = \frac{q_1 - q_2}{2}, \quad p_x = p_1 - p_2, \qquad y = \frac{q_1 + q_2}{2}, \quad p_y = p_1 + p_2.$$

The second integrals of motion I_2 (2.1) are the second order polynomials in momenta. The third independent integrals K are calculated by (2.7), where variables

$$(A) \quad \mu_1 = p_1 \quad \mu_2 = q_2 p_2\,, \qquad (B-C) \quad \mu_1 = q_1 p_1\,, \quad \mu_2 = p_2\,.$$

determine the left hand side of the canonical algebraic curves (2.4). At the case (D) the variables

$$(D) \qquad \mu_1 = p_2 q_1 - \frac{p_1 q_2}{3}\,, \qquad \mu_2 = p_2 q_2$$

have not such natural algebro-geometric meaning.

Generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$(A-B)\qquad N=\frac{I_2}{4\sqrt{-2\alpha}}\,,\qquad a=J+4\sqrt{-2\alpha}\,K\,,\qquad a^\dagger=J-4\sqrt{-2\alpha}\,K\,,$$

such that

$$\begin{aligned}
(A)\qquad a\,a^\dagger &= 4(4\alpha(2I_2+H)+\beta^2)\Big((2I_2-H)^2-64\alpha\gamma\Big)\,,\\
\{a,a^\dagger\} &= -128\alpha\sqrt{-2\alpha}\,(2I_2-H)(6I_2+H+\frac{\beta^2}{2\alpha})-64\alpha\gamma)\,,\\
(B)\qquad a\,a^\dagger &= \Big((2I_2+H)^2-64\alpha\beta\Big)\Big((2I_2-H)^2-64\alpha\gamma\Big)\,,\\
\{a,a^\dagger\} &= -16\sqrt{-2\alpha}\,\Big(\big((2I_2-H)^2-64\alpha\gamma\big)(2I_2+H)\\
&\qquad +\big((2I_2+H)^2-64\alpha\beta\big)(2I_2-H)\Big)\,.
\end{aligned}$$

At the two last cases we have

$$\begin{aligned}
(C)\qquad N &= \frac{I_2}{2\sqrt{-2\alpha}}\,,\qquad a=J+2\sqrt{-2\alpha}\,K\,,\qquad a^\dagger=J-2\sqrt{-2\alpha}\,K\,,\\
a\,a^\dagger &= \Big((2I_2-H)^2-2\alpha\beta\Big)\Big((2I_2+H)^2+2\alpha\beta\Big)\,,\\
\{a,a^\dagger\} &= -8\sqrt{-2\alpha}\Big(\big((2I_2-H)^2-2\alpha\beta\big)(2I_2+H)\\
&\qquad +\big((2I_2+H)^2+2\alpha\beta\big)(2I_2-H)\Big)\,,
\end{aligned}$$

and

$$\begin{aligned}
(D)\qquad N &= \frac{I_2}{6\sqrt{-2\alpha}}\,,\qquad a=J+6\sqrt{-2\alpha}\,K\,,\qquad a^\dagger=J-6\sqrt{-2\alpha}\,K\,,\\
a\,a^\dagger &= -4(2I_2-H)^3\,(2I_2+H)\,,\\
\{a,a^\dagger\} &= -96\sqrt{-2\alpha}\,(2I_2-H)^2\,(4I_2+H)\,.
\end{aligned}$$

At the case (D) (4.5) the quantum counterpart of this qubic deformed oscillator algebra has been used to study of the corresponding quantum superintegrable system [1].

4.2 Parabolic coordinates, cases F-H

Let us introduce two Stäckel matrices

$$\mathbf{S}_{F,G}=\begin{pmatrix}1 & 1\\ \\ q_1^{-1} & q_2^{-1}\end{pmatrix},\qquad \mathbf{S}_H=\begin{pmatrix}q_1^2 & q_2^2\\ \\ 1 & 1\end{pmatrix},\tag{4.15}$$

and take the following potentials

$$\begin{aligned}
&(F)\qquad U_1=-\frac{\alpha}{2q_1}-\frac{\beta}{2q_1^2},\qquad U_2=\frac{\alpha}{2q_2}-\frac{\gamma}{2q_2^2},\\
&(G)\qquad U_1=\frac{\alpha}{2q_1}-\frac{\beta}{4q_1^2},\qquad U_2=-\frac{\alpha}{2q_2}+\frac{\beta}{4q_2^2},\\
&(H)\qquad U_1=-4\alpha-8\sqrt{2}\beta\,q_1,\qquad U_2=4\alpha+8\sqrt{-2}\gamma\,q_2.
\end{aligned}\tag{4.16}$$

The corresponding Hamilton functions I_1 (2.1) coincide with the Hamilton functions H (4.1,4.3) and (4.4) after the following canonical point transformations

$$(F,G)\qquad x=q_1+q_2\,,\quad p_x=\frac{p_1q_2-p_2q_2}{q_1-q_2}\,,\quad y=2\sqrt{-q_1q_2}\,,\quad p_y=\frac{(p_1-p_2)\sqrt{-q_1q_2}}{q_1-q_2}\,,$$

and

$$(H)\qquad x=q_1^2+q_2^2\,,\quad p_x=\frac{1}{2}\frac{p_1q_1-p_2q_2}{q_1^2-q_2^2}\,,\quad y=2iq_1q_2\,,\quad p_y=\frac{i}{2}\frac{p_1q_2-p_2q_1}{q_1^2-q_2^2}\,.$$

The second integrals of motion I_2 (2.1) are the second order polynomials in momenta. The third independent integrals K are calculated by (2.7), where variables

$$(F,G)\quad \mu_1=q_1p_1\quad \mu_2=q_2p_2\,,\qquad (H)\quad \mu_1=p_1\,,\quad \mu_2=p_2\,,$$

define the left hand side of the canonical algebraic curves (2.4).

At all these cases (F,G) and (H) the generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$(F-H)\qquad N=\frac{I_2}{2\sqrt{H}}\,,\qquad a=J+2\sqrt{H}\,K\,,\qquad a^\dagger=J-2\sqrt{H}\,K\,,$$

such that

$$\begin{aligned}
(F)\qquad a\,a^\dagger &= \frac{1}{16}\Bigl(8\beta\,H-(2I_2+\alpha)^2\Bigr)\Bigl(8\gamma\,H-(2I_2-\alpha)^2\Bigr)\,,\\
\{a,a^\dagger\} &= -2\sqrt{H}\,\Bigl(I_2(2I_2+\alpha)(2I_2-\alpha)-2H\Bigl(\beta(2I_2-\alpha)+\gamma(2I_2+\alpha)\Bigr)\Bigr)\,,\\
(G)\qquad a\,a^\dagger &= -\frac{1}{16}\left(4\beta\,H+(2I_2+\alpha)^2\right)\left(4\beta\,H-(2I_2-\alpha)^2\right),\\
\{a,a^\dagger\} &= 2\sqrt{H}\left(I_2\,(2I_2+\alpha)(2I_2-\alpha)-2\alpha\beta\,H\right),\\
(H)\qquad a\,a^\dagger &= 16\Bigl(H\,(I_2-4\alpha)+32\gamma^2\Bigr)\Bigl(H\,(I_2+4\alpha)-32\beta^2\Bigr)\,,\\
\{a,a^\dagger\} &= -64H^{3/2}\,(H\,I_2-16\beta^2+16\gamma^2)\,.
\end{aligned}$$

4.3 Polar coordinates, case I

Let us introduce Stäckel matrix

$$\mathbf{S}_I=\begin{pmatrix}1&0\\ q_1^{-2}&1\end{pmatrix}\,,\tag{4.17}$$

and take the following potentials

$$(I)\qquad U_1=\alpha+\frac{\beta}{q_1},\qquad U_2=\frac{\gamma\,\cos(q_2)-\delta\,\sin(q_2)+\rho}{\Bigl(\delta\,\cos(q_2)+\gamma\,\sin(q_2)\Bigr)^2},\tag{4.18}$$

The corresponding Hamilton function I_1 (2.1) coincides with the Hamilton function H (4.9) if $q_1=r$ and $q_2=\phi$ are the standard polar coordinates on the plane. The second integrals of motion I_2 (2.1) are the second order polynomials in momenta. The third independent integrals K are calculated by (2.7), where variables

$$(I)\quad \mu_1=p_1\,,\qquad \mu_2=p_2\Bigl(\delta\,\cos(q_2)+\gamma\,\sin(q_2)\Bigr)\,,$$

define the canonical algebraic curves (2.4).

At $\delta = 1$ and $\gamma = 0$ the generators and defining relations of the nonlinear algebra of integrals (2.8) look like

$$
\begin{aligned}
(I) \qquad N &= -\sqrt{-I_2}\,, \qquad a = J + 2\sqrt{-I_2}\,K\,, \qquad a^\dagger = J - 2\sqrt{-I_2}\,K\,, \\
a\,a^\dagger &= (4\,H\,I_2 - 4\,\alpha\,I_2 + \beta^2)\,(4\,I_2^2 - 4\,\rho\,I_2 + 1)\,, \\
\{a, a^\dagger\} &= -8\,\sqrt{-I_2}\,\Big((2\,I_2 - \rho)\,\beta^2 + (12I_2^2 - 8\rho\,I_2 + 1)\,(H - \alpha)\Big)\,.
\end{aligned}
\tag{4.19}
$$

This system does not contain in the list of the known integrable systems [4, 6]. At the cases (I) (4.9) and (M) (4.12) we have common leading part P of the qubic integrals K. However, at the case (M) we can not rewrite the qubic integral in the "generalised angular momentum" form.

5 The Lax representation

In [11, 12] we proposed some construction of the 2×2 Lax matrices for the Stäckel systems with homogeneous Stäckel matrices [12] and with uniform potentials $U_j = U$. The Drach-Stäckel systems fall out from this subset of the Stäckel systems. Nevertheless, we could construct the 2×2 Lax matrices for these systems by using various covering [11] of the initial spheres $\mathcal{C}_{1,2}$ (2.6).

Here we consider the 4×4 Lax matrices for some Drach systems by using canonical transformations of the extended phase space, which induce transformations of the Lax matrices [12, 14]. Recall, if the Stäckel matrices $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ be distinguished the first row only, the corresponding Stäckel systems are related by canonical change of the time $q_{n+1} = t$ and conjugated momenta $p_{n+1} = -H$ [12]

$$
t \mapsto \widetilde{t}\,, \qquad d\widetilde{t} = \frac{\det\widetilde{\mathbf{S}}}{\det\mathbf{S}}\,dt\,, \qquad H \quad\mapsto\quad \widetilde{H} = \frac{\det\mathbf{S}}{\det\widetilde{\mathbf{S}}}\,H\,. \tag{5.1}
$$

Thus, starting with the Stäckel systems related with matrix $\mathbf{S} = \mathbf{S}_{cg}$ (3.12) we can study systems associated with matrices $\widetilde{\mathbf{S}} = \mathbf{S}_{be}$ (3.11) and $\widetilde{\mathbf{S}} = \mathbf{S}_k$ (3.15). Here subscripts mean the type of the Stäckel matrices for the different Drach systems.

The Stäckel systems with the constant matrix $\mathbf{S}_{cg}$ possess the following 4×4 Lax matrices [16, 11]

$$
\mathcal{L}(\lambda) = \begin{pmatrix} L_1(\lambda, p_1, q_1) & 0 \\ & \\ 0 & L_2(\lambda, p_2, q_2) \end{pmatrix}, \tag{5.2}
$$

with independent 2×2 non-trivial blocks $L_j(\lambda)$. For instance, two standard blocks may be chosen

$$
L_j(\lambda) = \begin{pmatrix} p_j & \lambda - q_j \\ -\left[\dfrac{\phi_j}{\lambda - q_j}\right]_{MN} & -p_j \end{pmatrix}, \qquad
L_j(\lambda) = \begin{pmatrix} \dfrac{p_j q_j}{\lambda} & 1 - \dfrac{q_j^2}{\lambda} \\ \dfrac{p_j^2}{\lambda} - \left[\dfrac{\phi_j}{1 - \frac{q_j^2}{\lambda}}\right]_{MN} & -\dfrac{p_j q_j}{\lambda} \end{pmatrix}.
$$

Here $\phi(\lambda)$ is a parametric function on spectral parameter λ and $[\xi]_N$ is the linear combinations of the Laurent projections [11].

According to [12, 14], canonical transformations of the extended phase space induce shift of the Lax matrices depending on the Hamilton function. Thus, by using one known Lax matrix $\mathcal{L}(\lambda)$ (5.2) we can construct another Lax matrices. Namely, canonical transformations of the time (5.1) give rise the following shift of the corresponding Lax matrices

$$
\widetilde{\mathcal{L}}(\lambda) = \mathcal{L}(\lambda) - \widetilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{pmatrix}, \qquad a, b = \pm1 \text{ or } \pm i\,, \tag{5.3}
$$

where values of the constants a, b depend on the chosen form of the blocks $L_j(\lambda)$.

Below we present some Lax matrices constructed by designated above scheme. In the case (c) the Lax matrix is given by

$$\mathcal{L}_c(\lambda) = \begin{pmatrix} p_1 & \lambda - q_1 & 0 & 0 \\ (\lambda + q_1)\left(-\dfrac{\alpha}{4} + \gamma q_1^2 \lambda^2\right) & -p_1 & 0 & 0 \\ 0 & 0 & ip_2 & i(\lambda - q_2) \\ 0 & 0 & i(\lambda + q_2)\left(-\dfrac{\alpha}{4} - \beta q_2^2 \lambda^2\right) & -ip_2 \end{pmatrix},$$

so the spectral curve

$$\Gamma(\lambda, \mu): \qquad \det\left(\mathcal{L}_c(\lambda) - \mu I\right) = 0$$

is a product

$$\left(\mu^2 - \frac{I_1}{2} - I_2 + \frac{\alpha\lambda^2}{4} + \frac{\gamma}{\lambda^2}\right)\left(\mu^2 - \frac{I_1}{2} + I_2 - \frac{\alpha\lambda^2}{4} + \frac{\beta}{\lambda^2}\right) = 0\,,$$

of the corresponding canonical Stäckel curves (2.6).

In the case (k) the Lax matrix is given by

$$\widetilde{\mathcal{L}}_k = \begin{pmatrix} p_1 & \lambda - q_1 & 0 & 0 \\ -\dfrac{\gamma(\lambda + q_1)}{2} - \beta & -p_1 & 0 & 0 \\ 0 & 0 & ip_2 & i(\lambda + q_2) \\ 0 & 0 & \dfrac{i\gamma(\lambda - q_2)}{2} + i\beta & -ip_2 \end{pmatrix} + \widetilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix},$$

where $\widetilde{H} = I_1$ be the Hamilton function (3.9). As in the previous example the spectral curve

$$\left(\mu^2 - \frac{\gamma\lambda^2}{2} + (\beta + I_1)\lambda + \alpha + I_2\right)\left(\mu^2 + \frac{\gamma\lambda^2}{2} + (\beta - I_1)\lambda + \alpha - I_2\right) = 0\,,$$

is a product of the corresponding Stäckel curves (2.6).

In the case (b) the Lax matrix is given by

$$\widetilde{\mathcal{L}}_b = \begin{pmatrix} \dfrac{p_1 q_1}{\lambda} & 1 - \dfrac{q_1^2}{\lambda} & 0 & 0 \\ \dfrac{p_1^2 - (\beta - 2\gamma)q_1^{-2}}{\lambda} & -\dfrac{p_1 q_1}{\lambda} & 0 & 0 \\ 0 & 0 & \dfrac{p_2 q_2}{\lambda} & 1 + \dfrac{q_2^2}{\lambda} \\ 0 & 0 & \dfrac{-p_2^2 + (\beta + 2\gamma)q_2^{-2}}{\lambda} & -\dfrac{p_2 q_2}{\lambda} \end{pmatrix} + \widetilde{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $\widetilde{H} = I_1$ be the Hamilton function (3.2). The corresponding spectral curve

$$\Gamma(y, \mu): \qquad \det\left(\widetilde{\mathcal{L}}_b(\lambda) - yI\right) = 0$$

is a product

$$\left(y^2 - I_1 + \frac{2\alpha + I_2}{\lambda} - \frac{\beta + 2\gamma}{\lambda^2}\right)\left(y^2 - I_1 + \frac{2\alpha - I_2}{\lambda} - \frac{\beta - 2\gamma}{\lambda^2}\right) = 0\,,$$

of the initial Stäckel curves (2.3), which could be rewritten in the canonical form (2.6).

All the spectral curves of these 4×4 Lax matrices $\mathcal{L}_\rfloor$, $\widetilde{\mathcal{L}}_k$ and $\widetilde{\mathcal{L}}_b$ give rise to the quadratic Stäckel integrals $I_{1,2}$ (2.1). The third integral K (2.7) may be extracted from the same matrices by using multivariable universal enveloping algebras [11]. In fact, this integral is a coefficient of the following multivariable polynomial

$$\mathrm{tr} P_\pi \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2) \otimes \mathcal{L}(\lambda_3) \otimes \mathcal{L}(\lambda_4)\,, \tag{5.4}$$

where P_π be permutation operator of auxiliary spaces corresponding to a Young diagram π [11]. The nonlinear algebra of integrals (2.8) may be reproduced by using the Poisson bracket relations between the Lax matrices $\otimes_j^k \mathcal{L}(\lambda_j)$ [11]. The formulae (5.4) for the 256×256 matrices has been proved by using the computer algebra system *Maple V*.

6 Conclusion

Let us discuss the list of all the known integrable natural Hamiltonian systems in the plane with a qubic integral [2, 4, 6]. We suppose that all these systems may be embedded into the family of the Stäckel systems [12], either into the subset of the generalized Stäckel systems [14] or these systems may be related to the Toda lattices and the Calogero-Moser systems [4, 14]. As an example the last case (l) (3.10) of the Drach systems and the Fokas-Lagerstrom [3] model belong to the generalized Stäckel systems [14]. The complete classification will be presented in the forthcoming publication.

In this note we proved this proposition for all the Drach systems [2]. Moreover, we rewrite the qubic integrals for the superintegrable Drach systems in common form (2.7). This generalized "angular momentum" may be used to construct another n-dimensional superintegrable Stäckel systems with the cubic integrals of motion. For instance, let us consider the Hamilton function

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{\gamma + \delta}{r} + \frac{1}{x^2 + y^2}\left(\frac{\alpha\,(r - z)}{r} + \frac{\beta(r + z)}{r} + U(y/x)\right), \tag{6.5}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $U(y/x)$ be arbitrary function. The corresponding equations of motion are separable in the parabolic coordinates

$$q_1 = r + z\,, \qquad q_2 = r - z\,, \qquad q_3 = \arctan(y/x)\,,$$

which related to the following Stäckel matrix

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 \\ q_1^{-1} & q_2^{-1} & 0 \\ q_1^{-2} & q_2^{-2} & -4 \end{pmatrix}.$$

The Hamilton function (6.5) coincides with the Stäckel integral I_1 (2.1) if

$$U_1 = \frac{\alpha}{q_1^2} + \frac{\gamma}{q_1}\,, \qquad U_2 = \frac{\beta}{q_2^2} + \frac{\delta}{q_2}\,, \qquad U_3 = U(q_3)\,.$$

Thus we have integrable Stäckel system with the independent integrals of motion $I_{1,2}$ and I_3, which are quadratic polynomials in momenta.

Canonical algebraic curves is defined in variables

$$\mu_1 = p_1\, q_1\,, \qquad \mu_2 = p_2\, q_2\,, \qquad \mu_3 = p_3\,.$$

To substitute these variables in the "generalized angular momentum" (2.7) one gets additional qubic in momenta integral of motion K. In the initial physical variables this integral K looks like

$$\begin{aligned} K &= (x^2 + y^2)\, p_z^3 - 2z\, p_z^2\, (p_x\, x + p_y\, y) - p_z\, (p_x\, x + p_y\, y)^2 \\ &+ r^2 \left(\frac{\partial H}{\partial z}\, (p_x\, x + p_y\, y) + p_z \left(p_x^2 + p_y^2 - x\frac{\partial H}{\partial x} - y\frac{\partial H}{\partial y} \right) \right). \end{aligned}$$

It will be interesting to understand the algebro-geometric origin of this "generalized angular momentum" (2.7).

References

[1] D. Bonatos, C. Daskaloyannis, and K. Kokkotas. *Phys.Rev.A.*, v.50, p.3700, 1994.

[2] J. Drach. *Comptes Rendus*, (Paris), v.200, p.22, 1935.

[3] A.S. Fokas, P. Lagerstrom. J.Math.Ann.Appl., v.74, p.325, 1980.

[4] J. Hietarinta. *Phys.Rep.*, v.147, p.87, 1987.

[5] C.R. Holt. *J.Math.Phys.*, v.23, p.37, 1982.

[6] M. Karlovini, K. Rosquist *A unified treatment of cubic invariants at fixed and arbitrary energy.* Preprint solv-int/9811005, 1998.

[7] P. Letourneau and L. Vinet. *Annals of Physics*, v.243, p.144, 1995.

[8] P. Stäckel. *Comptes Rendus*, (Paris), v.116, p.485 and p.1284, 1893.

[9] G. Thompson. *J.Math.Phys*, v.25, p.3474, 1984.

[10] A.V. Tsiganov. *J.Phys.A.* v.31, p.2075, 1998.

[11] A.V. Tsiganov. *J.Math.Phys.* v.40, p.279, 1999.

[12] A.V. Tsiganov. *J.Phys.A.*, v.32, p.7965, 1999.

[13] A.V. Tsiganov. *J.Phys.A.*, v.32, p.7983, 1999.

[14] A.V. Tsiganov. *Canonical transformations of the extended phase space, Toda lattices and Stäckel family of integrable systems.*, Preprint solv-int/9909006, 1999, Submitted to J.Phys A.

[15] A.V. Tsiganov. *Canonical transformations of the time for the Toda lattice and the Holt system.*, Preprint solv-int/99011001, 1999.

[16] P. Vanhaecke. *Math.Z.*, v.211, p.265, 1992.

[17] E.T. Whittaker. *Analytical dynamics.*, Cambridge University Press, Fourth edition, Cambridge, 1973.